Skew-symmetric schemes for stochastic differential equations with non-Lipschitz drift: an unadjusted Barker algorithm

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‘Skew26’, Padova, January 2026

UCL

Collaborators

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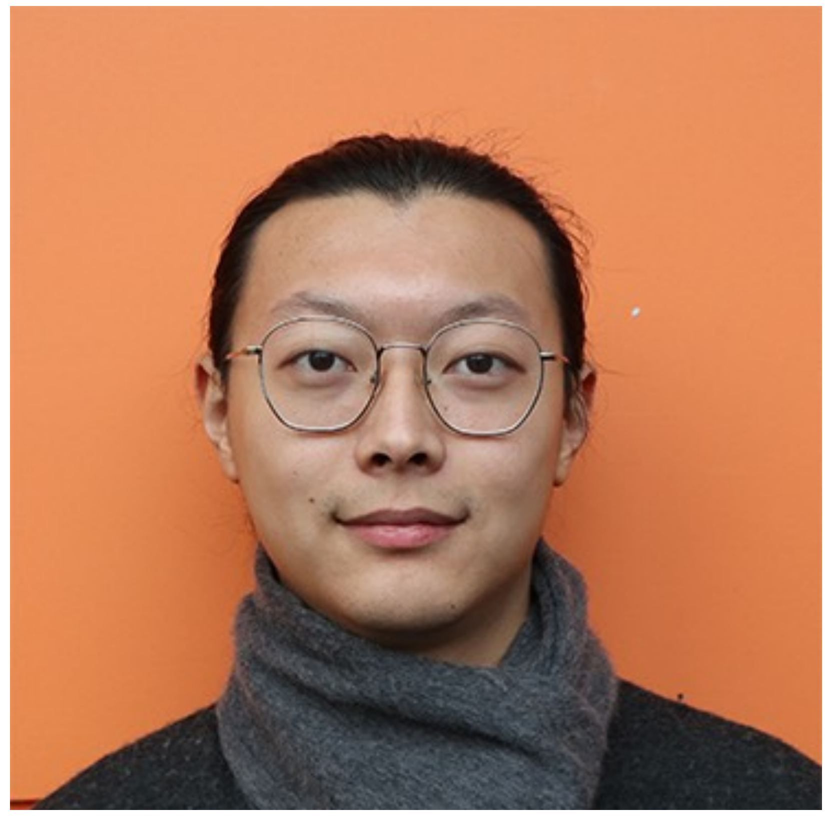

Ruiyang Zhang
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(Former UCL undergraduate)

(earlier work with Giacomo Zanella, Jure Vogrinc & Max Hird).

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  - How can we generate an approximate skeleton sample path $\{X_0, X_h, X_{2h}, \ldots\}$?
  - How can we approximate the transition density $p(x_0)f(x_h \mid x_0)f(x_{2h} \mid x_h)\ldots$ associated with this skeleton?

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- Statistical applications: sampling algorithms (e.g. Bayesian inference), parameter inference for diffusion processes, diffusion priors for Bayesian modelling

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- If desired, after each time step a Metropolis—Hastings filter can also be applied
- In Bayesian inference applications we can take π as the posterior.
    - NOTE! Only $\nabla U(x)$ needed, i.e. only unnormalised π needed (well-suited to Bayes)

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    - **Alternatives:** implicit schemes (expensive), tamed Euler, truncated Euler...

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- **Equilibrium sampling:** As $T \to \infty$ for fixed $h > 0$ the numerical process is typically a transient Markov chain (Roberts & Tweedie, 1996), i.e. $\forall$ probability distributions π_h, as $n \to \infty$

$$\|P_h^n(x, \cdot) - \pi_h\|_{TV} \nrightarrow 0$$

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- Using this intuition, it can be shown that as $N \to \infty$, $|X_{Nh}| \to \infty$ w.p. 1 from any starting point
- Taking $\pi(dy) \propto e^{-y^4/4}dy$, then this SDE is the corresponding overdamped Langevin diffusion

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- **Poisson Generalised Linear Model:** data $\{(z_i, y_i)\}_{i=1}^n$, $z_i \in \mathbb{R}^d$, $y_i \in \mathbb{Z}_+$, model

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- Setting $\pi_0(\beta) \propto e^{-\frac{1}{2}\beta^T A \beta}$ gives a posterior $\pi(\beta) \propto e^{-U(\beta)}$ with

$$\nabla U(\beta) = \sum_i (e^{z_i^T \beta} - y_i) z_i + A\beta$$

i.e. **exponential** in β

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- Here $\zeta_{nh} \sim \nu_{X_{nh},h}$ with $\nu_{x,h}$ a **skew-symmetric probability distribution**, having density

$$\nu_{x,h}(\zeta) = 2 \cdot p_h(x,\zeta) \cdot \phi_{x,h}(\zeta)$$

  - $\phi_{x,h}$ is a $N(0,h\sigma^2(x))$ density
  - $p_h(x,\zeta) \in [0,1]$ satisfies $p_h(x,\zeta) + p(x,-\zeta) = 1$ (e.g. a CDF of symmetric r.v.)

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- We stipulate the additional requirement that

$$\partial_\zeta p_h(x,\zeta)\,|_{\zeta=0} = \frac{\mu(x)}{2\sigma^2(x)}$$

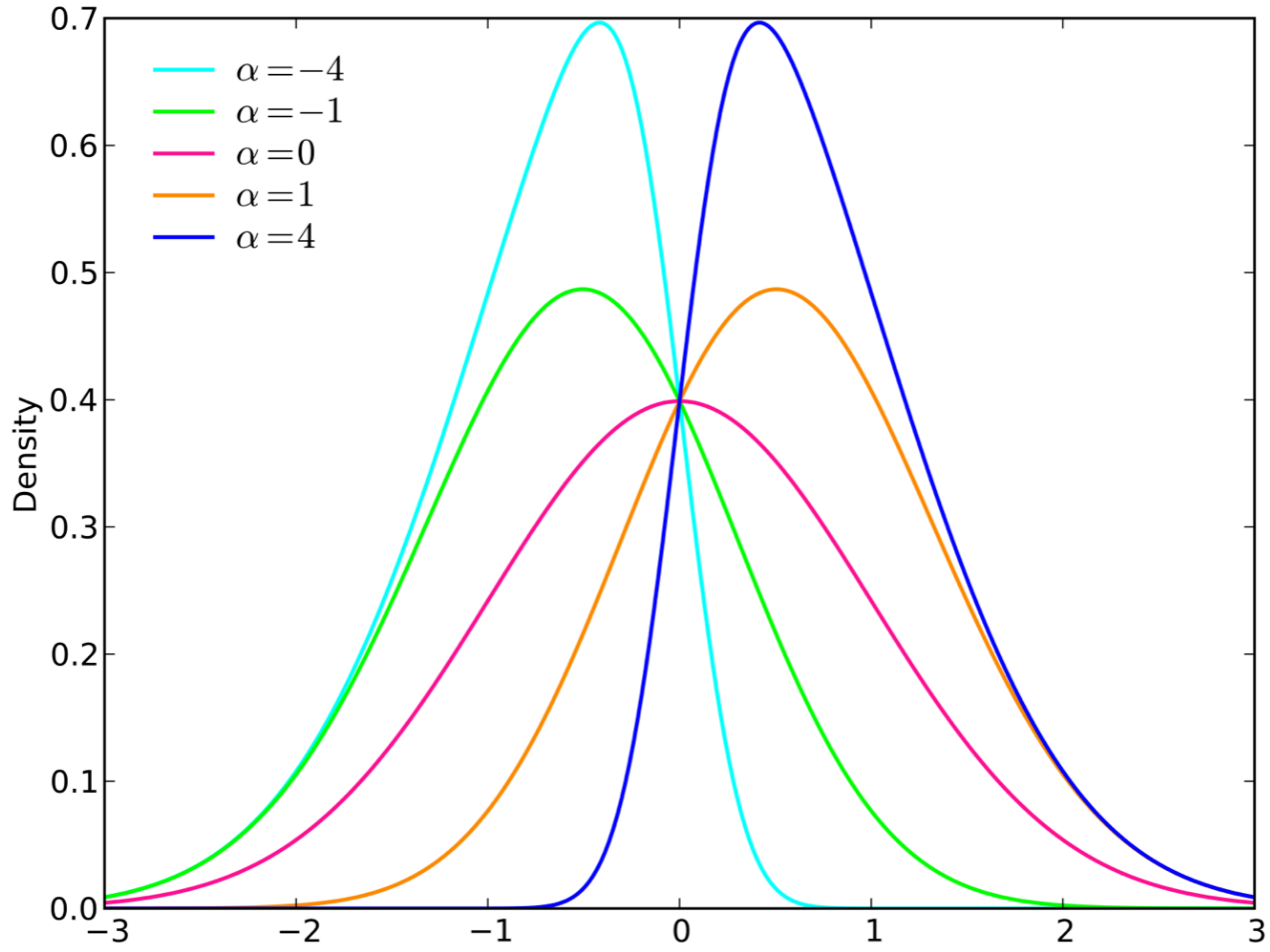
Example: skew-Normal distribution

(picture: 'Skew-Normal distribution' wikipedia page)

e.g. Skew-Normal
$s(z) = 2\Phi(\alpha z)\phi(z)$

The skew-Normal distribution is an example with explicit mean

$$\sqrt{\frac{2}{\pi}} \cdot \frac{\alpha}{\sqrt{1+\alpha^2}}$$



As $\alpha \to \infty$ density converges to **half-Normal**

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where $\xi_{nh} \sim N(0, h\sigma^2(X_{nh}))$ and

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- In words:

1. Simulate a mean-zero **Gaussian increment** ξ_{nh}
2. **Skew in the direction of the drift** $\mu(X_n)$ via multiplication by b_{nh}

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- Set $\mathbb{P}[b_{i,nh} = +1] = p_i(X_{nh}, \xi_{nh})$
- A simple choice of p_i is

$$p_i(x, \xi) = F(\alpha_{i,x} \cdot \xi_i)$$

where F is CDF of a symmetric r.v. and (e.g. for σ diagonal)

$$\alpha_{i,x} = \frac{\mu_i(x)}{2F'(0)\sigma_{ii}(x)}$$

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Result 1 (Weak Convergence): Under appropriate assumptions the skew-symmetric scheme is **weak order 1**, meaning as $h \to 0$, for $k \leq \lfloor T/h \rfloor$ and polynomially growing K and $f \in C^4(\mathbb{R}^d)$

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Remarks on assumptions & proof

- Drift μ is C^4 with polynomial growth, locally Lipschitz + satisfies *one-sided* Lipschitz condition

$$(\mu(x) - \mu(y)) \cdot (x - y) \leq C(1 + \|x - y\|^2)$$

- Volatility σ is C^4, Lipschitz
- Proof extends 'General convergence theorem' of Milstein & Tretyakov to non-Lipschitz case

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Remarks on assumptions & proof

- Additional regularity on μ to allow contraction, e.g.

$$\lim_{\|x\| \to \infty, x_i < -\|x_i\|_\infty/2} \mu_i(x) = \infty, \qquad \lim_{\|x\| \to \infty, x_i > \|x_i\|_\infty/2} \mu_i(x) = -\infty$$

- Bounded + diagonal volatility, $N \leq \sigma_{ii}(x) \leq M$, some (weak) regularity of F
- Proof is based on Foster—Lyapunov drift condition (e.g. Meyn & Tweedie)

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Remarks on assumptions & proof

- True diffusion process has equilibrium π, which has moments of all orders
- Slightly different type of contraction on drift

$$\langle x, \mu(x) \rangle \leq -C\|x\|^a$$

- Proof based on theorem of Lelièvre & Stoltz (expansion of numerical scheme generator)

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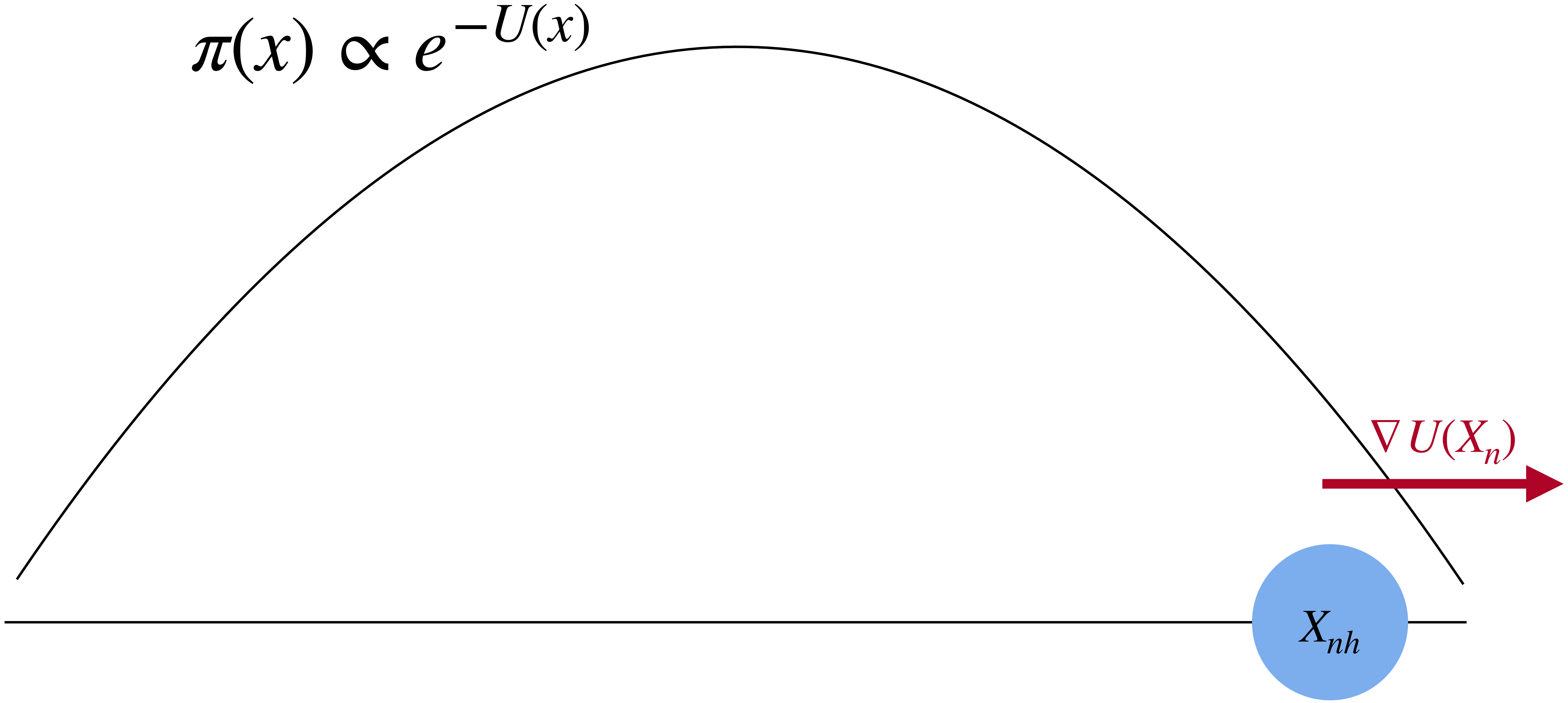
- Skew-symmetric scheme (using logistic CDF to skew) becomes $X_{i,(n+1)h} = X_{i,nh} + b_{i,nh} \cdot \xi_{i,nh}$ with

$$\mathbb{P}[b_{i,nh} = +1] = \frac{1}{1 + \exp\{\partial_i U(X_{nh}) \cdot \xi_{i,nh}\}}$$

Unadjusted Barker algorithm

Example on $\mathbb{R}$



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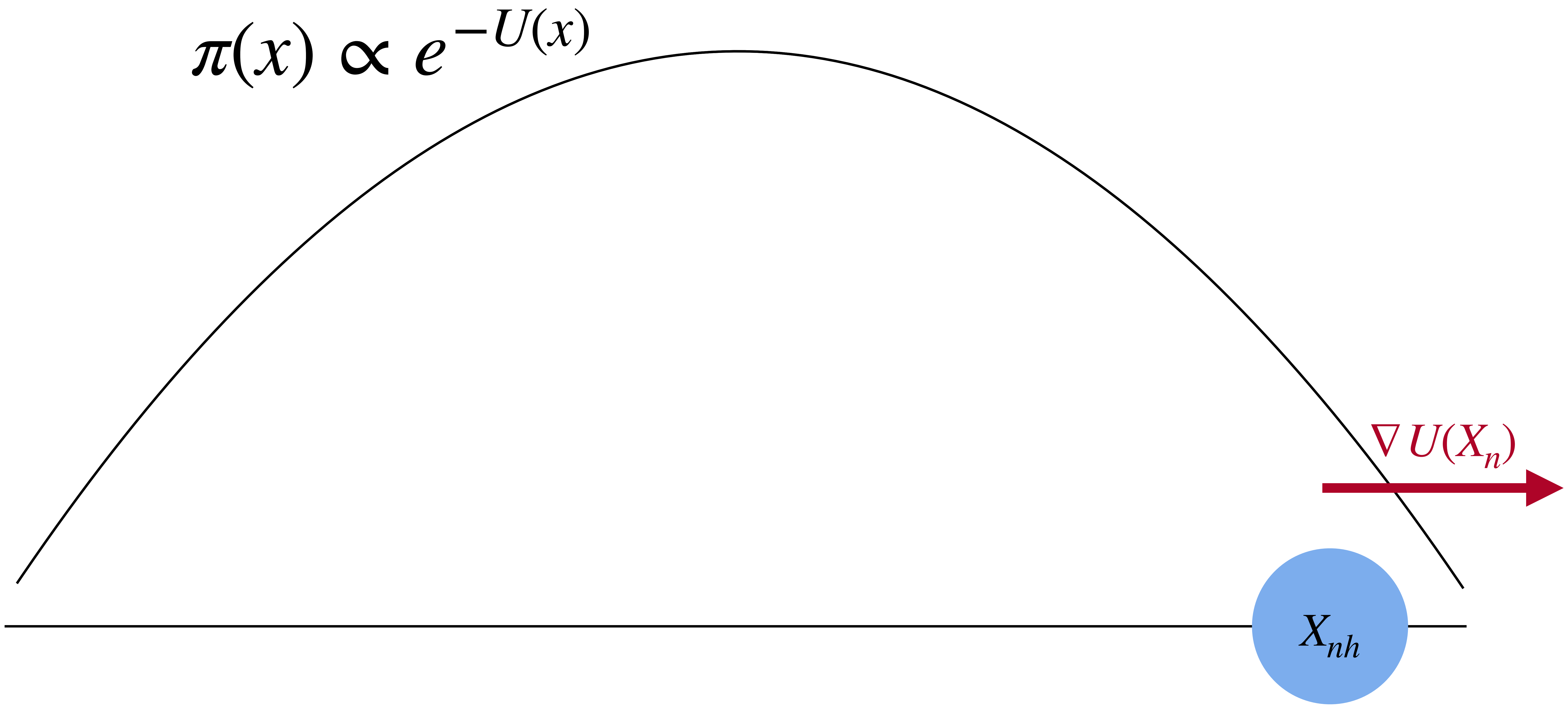

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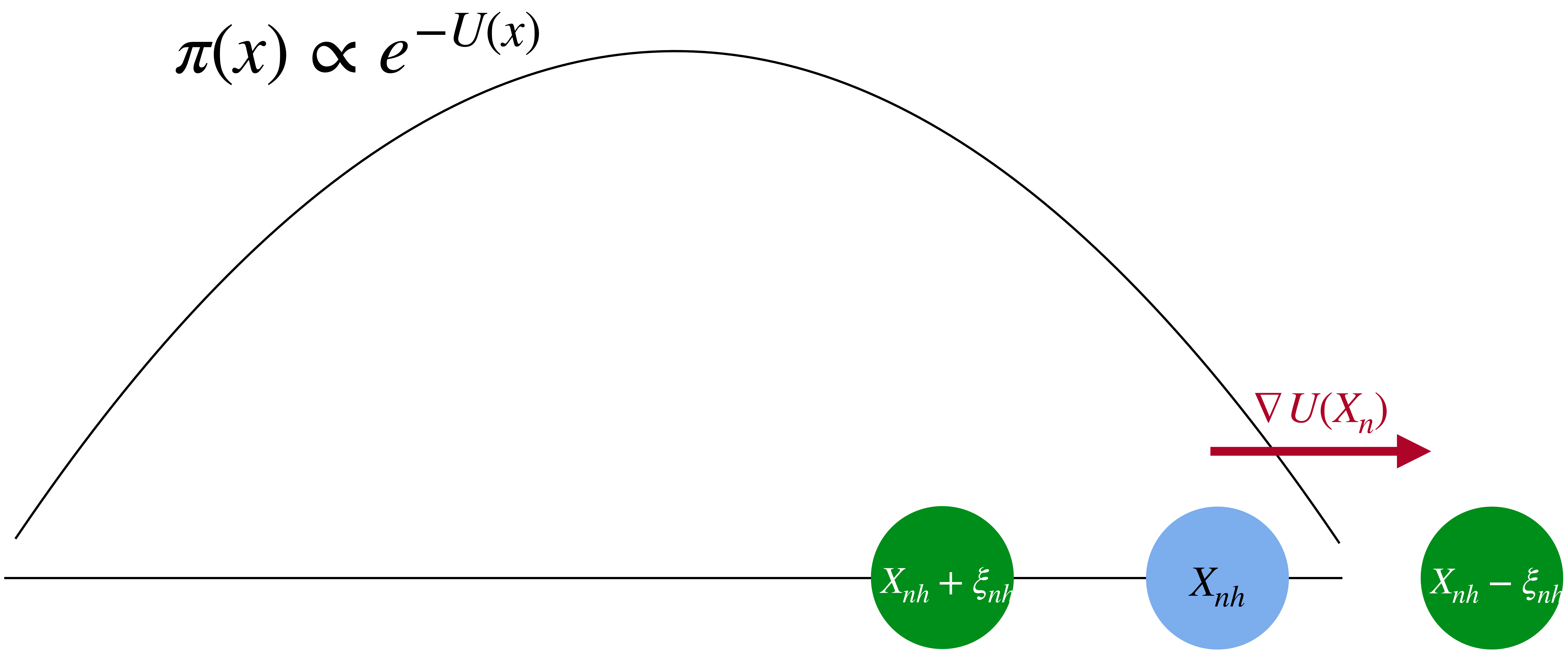

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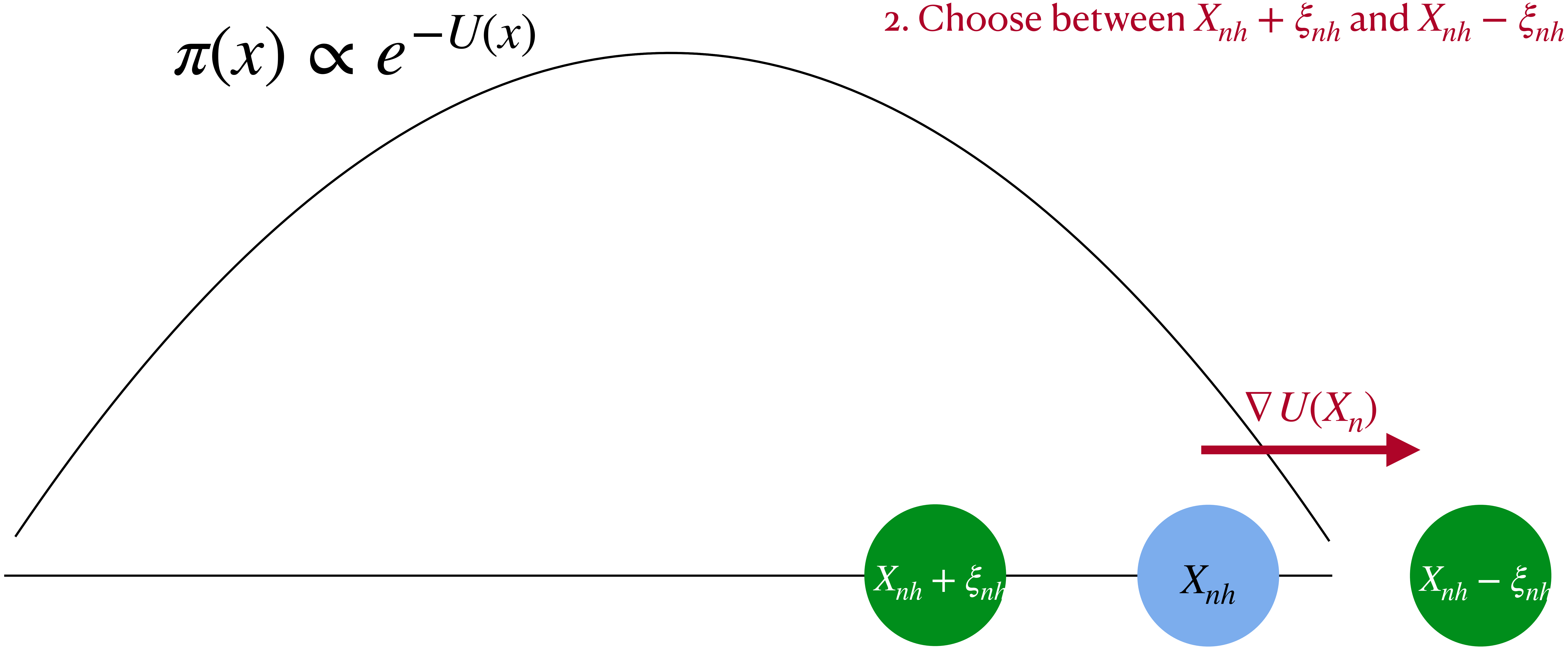
Example on $\mathbb{R}$

1. Draw $\xi_{nh} \sim N(0,2h)$

2. Choose between $X_{nh} + \xi_{nh}$ and $X_{nh} - \xi_{nh}$



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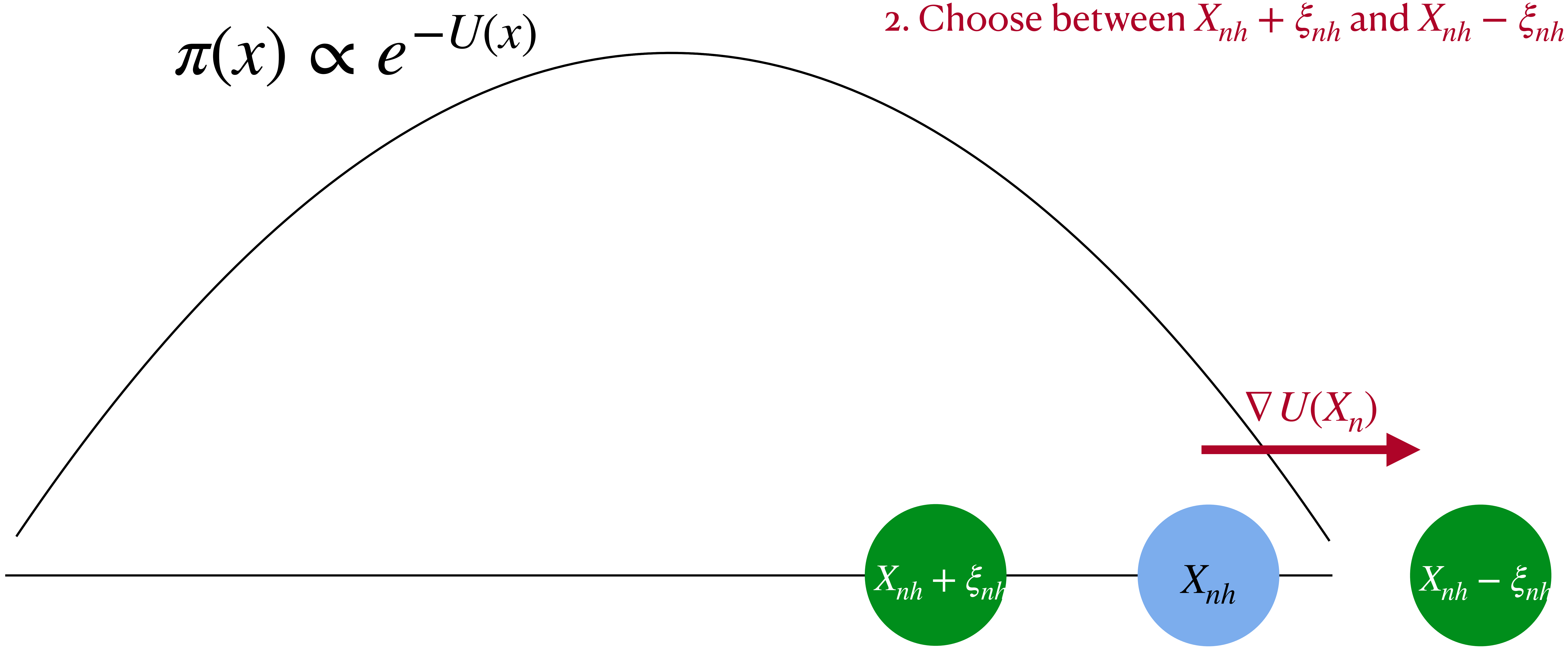

$p(X_{nh}, \xi_{nh}) > \frac{1}{2} \implies X_{nh} + \xi_{nh}$ the more likely step

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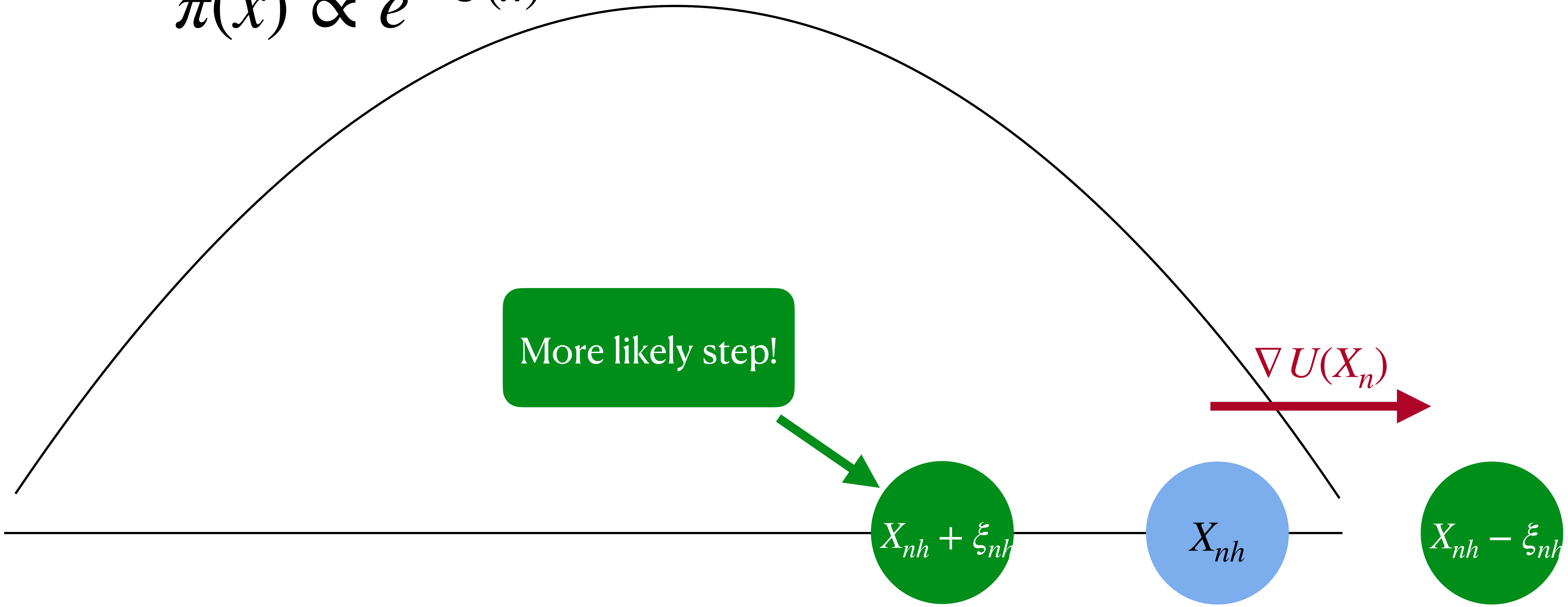
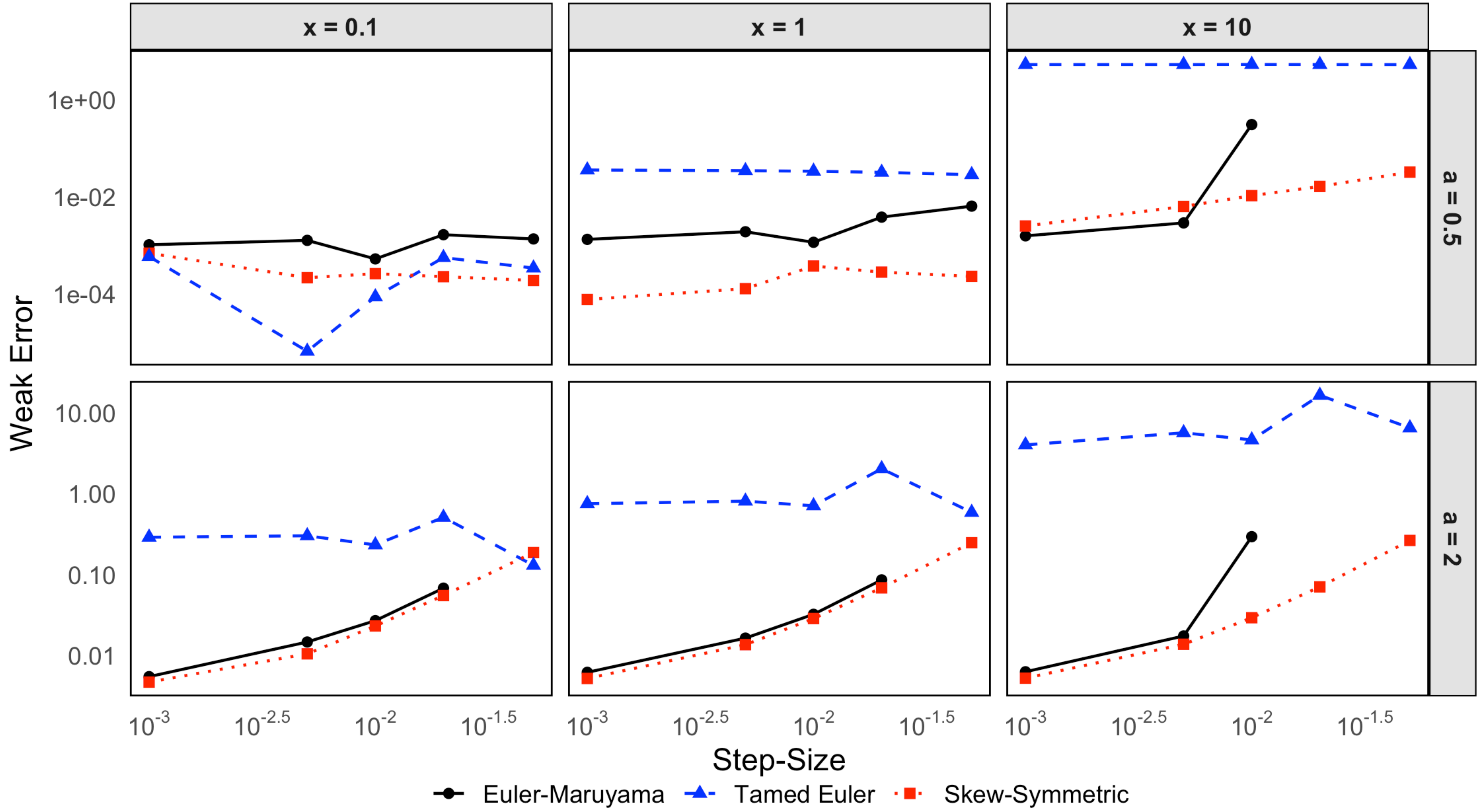
$$\pi(x) \propto e^{-U(x)}$$



$$p(X_{nh}, \xi_{nh}) > \frac{1}{2} \implies X_{nh} + \xi_{nh} \text{ the more likely step}$$

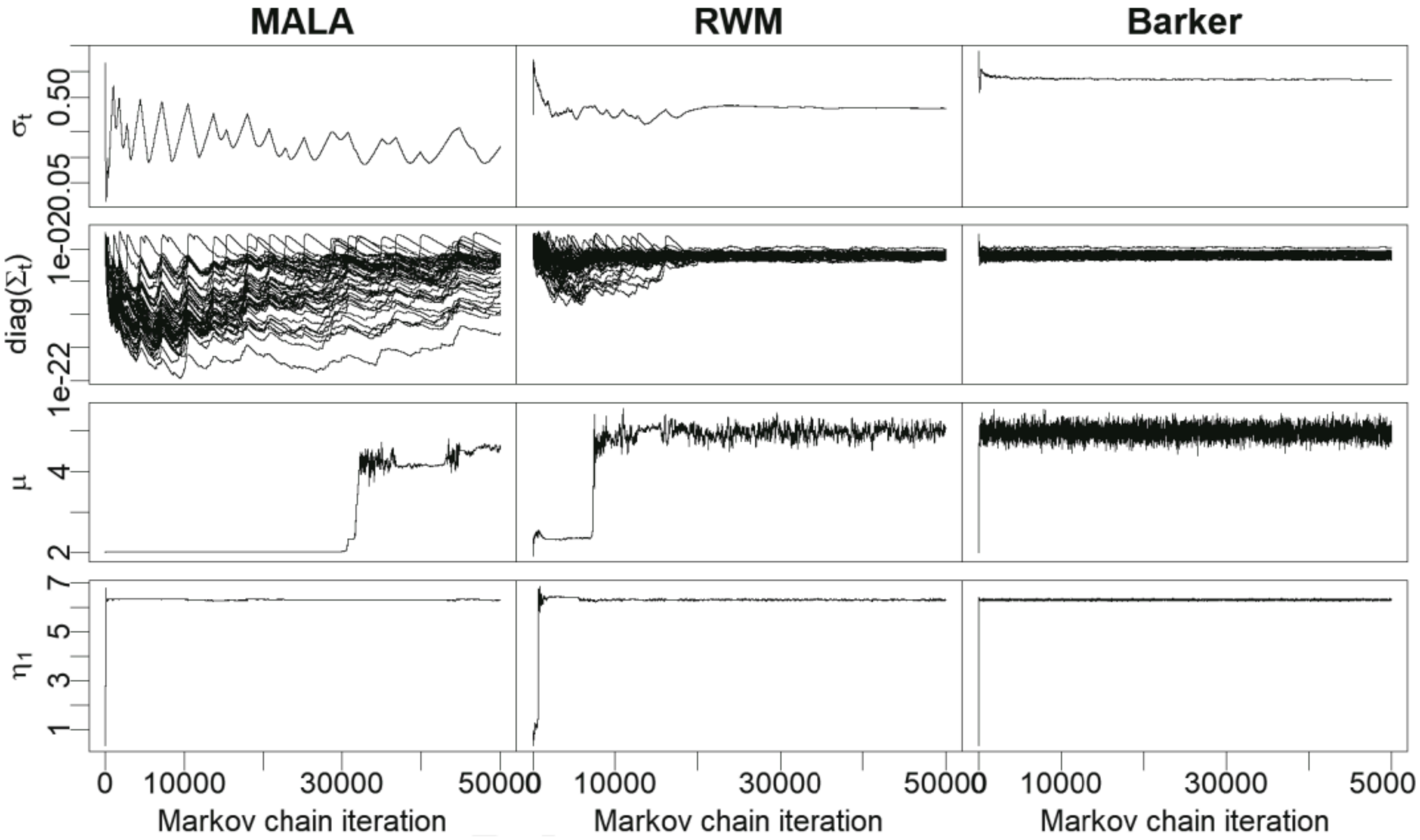
Example: finite time simulation (stochastic Ginzburg–Landau model)

$$dY_t = ((a^2/2)Y_t - Y_t^3)dt + aY_t dW_t, \quad Y_0 = x$$



Example: long time simulation (Metropolis-adjusted)

Poisson random effects model, adaptive step-size σ_t and diagonal preconditioned Σ_t



Discussion

- Current ongoing work (preprint coming in ~3 months)
  - Finalising strong mean-squared convergence/path-wise accuracy…multi-level Monte Carlo implementation
  - Hypo-elliptic scheme (e.g. kinetic Langevin sampling) + hybrid scheme (combined with Euler—Maruyama)
- Also check out the **rmcmc** R package for MCMC sampling with skew-symmetric distributions (via unadjusted/Metropolis-adjusted Barker algorithm) and general comparison of many different sampling algorithms (on CRAN)

- Further reading:

  - Iguchi, Y., Livingstone, S., Nüsken, N., Vasdekis, G., & Zhang, R. Y. (2026). Skew-symmetric schemes for stochastic differential equations with non-Lipschitz drift: an unadjusted Barker algorithm. *IMA Journal of Numerical Analysis (forthcoming)*. **(Unadjusted scheme)**
  - Livingstone, S., & Zanella, G. (2022). The Barker proposal: Combining robustness and efficiency in gradient-based MCMC. *Journal of the Royal Statistical Society Series B: Statistical Methodology, 84(2), 496-523*. **(Metropolis-adjusted scheme)**
  - Vogrinc, J., Livingstone, S., & Zanella, G. (2023). Optimal design of the Barker proposal and other locally balanced Metropolis–Hastings algorithms. *Biometrika, 110(3), 579-595*. **(Metropolis-adjusted scheme)**
  - Hardcastle, L., Livingstone, S., & Baio, G. (2025). Diffusion piecewise exponential models for survival extrapolation using Piecewise Deterministic Monte Carlo. arXiv preprint arXiv:2505.05932. **(Diffusion priors for Bayesian models)**

Appendices

Example 1: Weak order and equilibrium bias

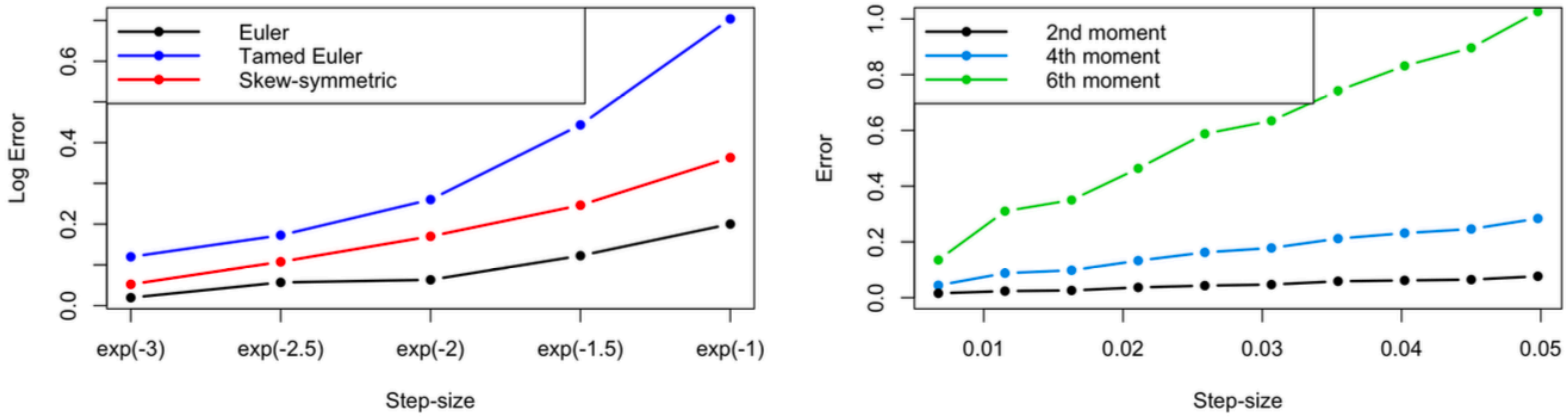


Figure 1: Figure on the left-hand side shows logarithm of the absolute error when computing $\mathbb{E}_x[Y_T^2]$ for T fixed for the Ornstein–Uhlenbeck process. Figure on the right-hand side shows absolute error when computing moments of the density $\pi(x) \propto e^{-x^4/4}$ by simulating the overdamped Langevin diffusion over long time scales.

Example 2: Poisson random effects model

$$
\begin{aligned}
y_{ij}|\eta_i &\sim \text{Poi}(e^{\eta_i}), && j = 1, \ldots, J, \\
\eta_i|\mu &\sim N(\mu, 1), && i = 1, \ldots, I, \\
\mu &\sim N(0, \sigma_\mu^2).
\end{aligned}
$$

$$\pi(x) \propto \exp\{-U(x)\}$$

$$U(x) = J\sum_i e^{\eta_i} - \sum_{i,j} y_{ij}\eta_i + \frac{1}{2}\sum_i(\eta_i - \mu)^2 + \frac{\mu^2}{2\sigma_\mu^2}.$$

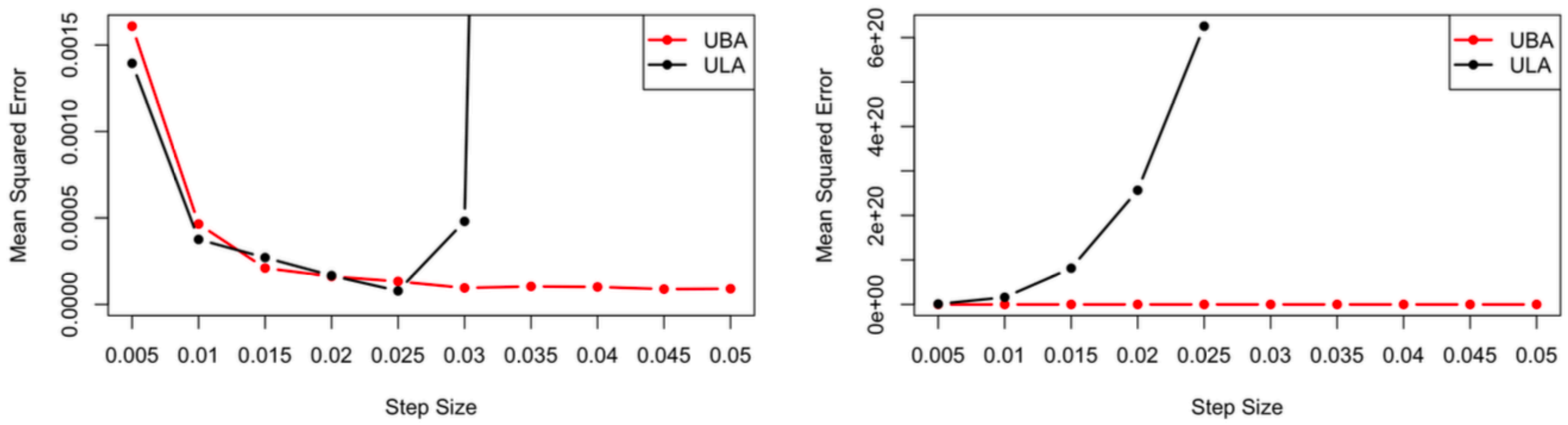


Figure 2: Mean squared error comparisons for the Poisson random effects example. Figure 2(a) (left-hand side) shows means squared error when initialised at the true value μ^*. Figure 2(b) (right-hand side) is initialised from a $N(5, 10^2)$ sample.

Example 3: Soft-spheres in an anharmonic trap

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- Particle locations $X_t^{(i)} \in \mathbb{R}^2$ for $i \in \{1,2,\ldots,N\}$ follow dynamics (for $B, A, D, r > 0$)

$$dX_t^{(i)} = \underbrace{-4BX_t^{(i)}\|X_t^{(i)}\|^2 dt}_{\text{anharmonic trap}} + \underbrace{\frac{A}{Nr^2}\sum_{j=1}^{N}(X_t^{(i)} - X_t^{(j)})e^{-\|X_t^{(i)}-X_t^{(j)}\|^2/2r^2}dt}_{\text{Soft-spheres interaction}} + \sqrt{2D}\,dW_t$$

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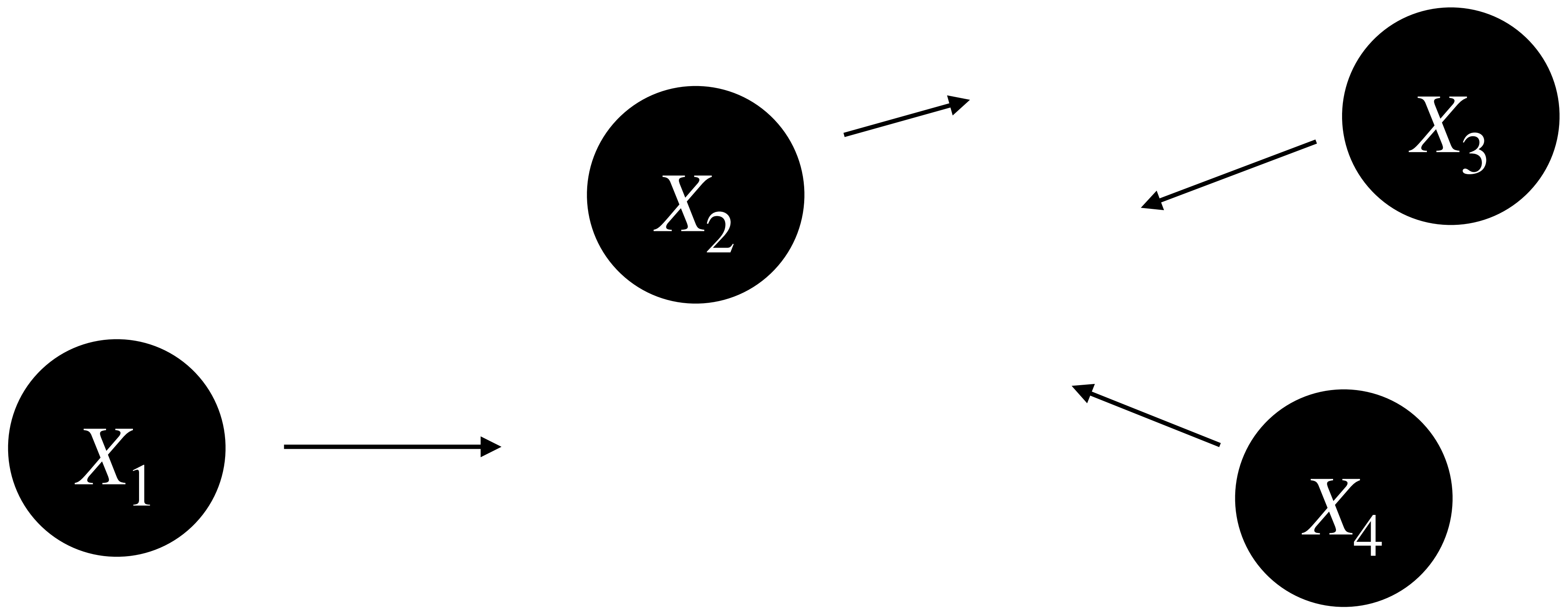

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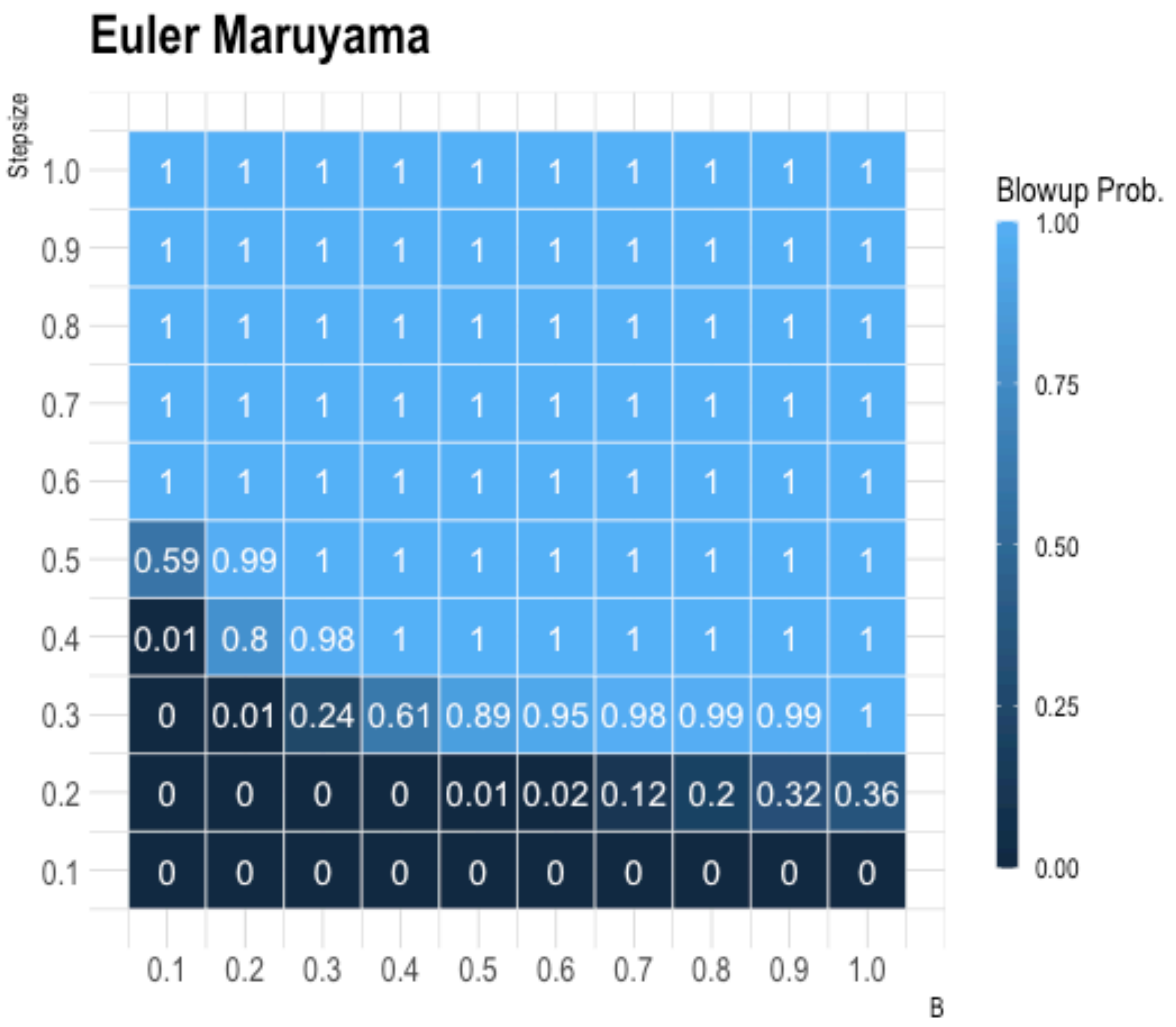
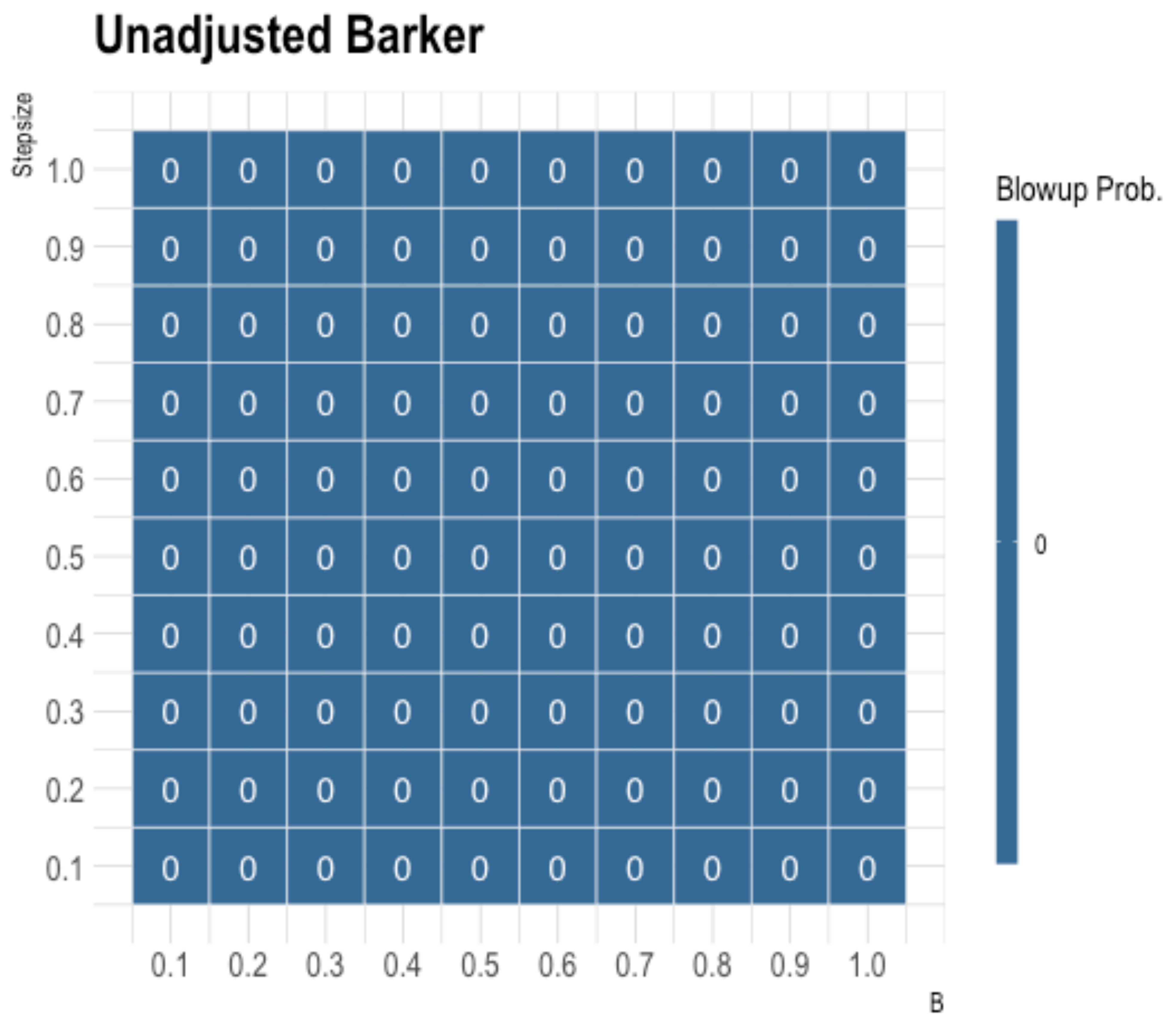
Numerical explosion probability after 10 steps, 100 runs, N=50 particles





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- When μ grows **super linearly** (i.e. not Lipschitz) the latter is unstable
- To compare with Tamed Euler choose e.g. $\sigma(x) = \max(\epsilon, |x|)$ (see preprint)